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Advanced Linear Algebra (MA 409)
Problem Sheet - 24

The Adjoint of a Linear Operator

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1. Label the following statements as true or false. Assume that the underlying inner product spaces are finite-dimensional.
   (a) Every linear operator has an adjoint.
   (b) Every linear operator on V has the form $x \to \langle x, y \rangle$ for some $y \in V$.
   (c) For every linear operator T on V and every ordered basis β for V, we have $[T^*]_\beta = ([T]_\beta)^*$.
   (d) The adjoint of a linear operator is unique.
   (e) For any linear operators T and U and scalars a and b.
   $$(aT + bU)^* = aT^* + bU^*.$$
   (f) For any $n \times n$ matrix A, we have $(L_A)^* = L_{A^*}$.
   (g) For any linear operator T, we have $(T^*)^* = T$.

2. For each of the following inner product spaces V (over F) and linear transformations $g : V \to F$, find a vector y such that $g(x) = \langle x, y \rangle$ for all $x \in V$.
   (a) $V = \mathbb{R}^3$, $g(a_1, a_2, a_3) = a_1 - 2a_2 + 4a_3$
   (b) $V = \mathbb{C}^2$, $g(z_1, z_2) = z_1 - 2z_2$
   (c) $V = P_2(\mathbb{R})$ with $\langle f, h \rangle = \int_0^1 f(t)h(t)\,dt, \quad g(f) = f(0) + f'(1)$

3. For each of the following inner product spaces V and linear operators T on V, evaluate T^* at the given vector in V.
   (a) $V = \mathbb{R}^2$, $T(a, b) = (2a + b, a - 3b)$, $x = (3, 5)$.
   (b) $V = \mathbb{C}^2$, $T(z_1, z_2) = (2z_1 + iz_2, (1 - i)z_1)$, $x = (3 - i, 1 + 2i)$.
   (c) $V = P_1(\mathbb{R})$ with $\langle f, g \rangle = \int_{-1}^1 f(t)g(t)\,dt, \quad T(f) = f' + 3f, f(t) = 4 - 2t.$

4. Using a matrix argument, prove the following for nonsquare $m \times n$ matrices A and B.
   (a) $(A + B)^* = A^* + B^*$;
   (b) $(cA)^* = \bar{c}A^*$ for all $c \in F$;
   (c) $(AB)^* = B^*A^*$;
   (d) $A^{**} = A$;
   (e) $I^* = I$.

5. Let T be a linear operator on an inner product space V. Let $U_1 = T + T^*$ and $U_2 = TT^*$. Prove that $U_1 = U_1^*$ and $U_2 = U_2^*$.

6. Give an example of a linear operator T on an inner product space V such that $N(T) \neq N(T^*)$.

7. Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Prove that if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

8. Prove that if $V = W \oplus W^\perp$ and T is the projection on W along $W^\perp$, then $T = T^*$.

   Hint: Recall that $N(T) = W^\perp$.

9. Let T be a linear operator on an inner product space V. Prove that $\|T(x)\| = \|x\|$ for all $x \in V$ if and only if $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all $x, y \in V$.

10. For a linear operator T on an inner product space V, prove that $T^*T = T_0$ implies $T = T_0$. Is the same result true if we assume that $TT^* = T_0$?

11. Let V be an inner product space, and let T be a linear operator on V. Prove the following results.

    (a) $R(T^*)^\perp = N(T)$.

    (b) If V is finite-dimensional, then $R(T^*) = N(T)^\perp$.

12. Let T be a linear operator on a finite-dimensional inner product space V. Prove the following results.

    (a) $N(T^*T) = N(T)$. Deduce that $rank(T^*T) = rank(T)$.

    (b) $rank(T) = rank(T^*)$. Deduce from (a) that $rank(TT^*) = rank(T)$.

    (c) For any $n \times n$ matrix A, $rank(A^*A) = rank(AA^*) = rank(A)$.

13. Let V be an inner product space, and let $y, z \in V$. Define $T : V \to V$ by

$$T(x) = \langle x, y \rangle z$$

    for all $x \in V$. First prove that T is linear. Then show that T^* exists, and find an explicit expression for it.

The following definition is used in Exercises 14-16 and is an extension of the definition of the adjoint of a linear operator.

Definition. Let $T : V \to W$ be a linear transformation, where V and W are finite-dimensional inner product spaces with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. A function $T^* : W \to V$ is called an **adjoint** of T if $\langle T(x), y \rangle_2 = \langle x, T^*(y) \rangle_1$ for all $x \in V$ and $y \in W$.

14. Let $T : V \to W$ be a linear transformation, where V and W are finite-dimensional inner product spaces with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. Prove the following results.

    (a) There is a unique adjoint T^* of T, and T^* is linear.

    (b) If β and γ are orthonormal bases for V and W, respectively, then $[T^*]_\gamma^\beta = ([T]_\beta^\gamma)^*$.

    (c) $rank(T^*) = rank(T)$.

    (d) $\langle T^*(x), y \rangle_1 = \langle x, T(y) \rangle_2$ for all $x \in W$ and $y \in V$.

    (e) For all $x \in V$, $T^*T(x) = 0$ if and only if $T(x) = 0$.

15. We now recall the result : Let V be an inner product space, and let T and U be linear operators on V. Then

    (a) $(T + U)^* = T^* + U^*$;

(b) $(cT)^* = \bar{c}T^*$ for any $c \in F$;

(c) $(TU)^* = U^*T^*$;

(d) $T^{**} = T$;

State and prove a result that extends the four parts (a)-(d) of the above result, using the preceding definition.

16. Let $T : V \to W$ be a linear transformation, where V and W are finite-dimensional inner product spaces. Prove that $(R(T^*))^\perp = N(T)$, using the preceding definition.

17. Let A be an $n \times n$ matrix. Prove that $\det(A^*) = \overline{\det(A)}$.

18. Suppose that A is an $m \times n$ matrix in which no two columns are identical. Prove that A^*A is a diagonal matrix if and only if every pair of columns of A is orthogonal.

19. For each of the sets of data that follows, use the least squares approximation to find the best fits with both (i) a linear function and (ii) a quadratic function. Compute the error E in both cases.

    (a) $\{(-3, 9), (-2, 6), (0, 2), (1, 1)\}$

    (b) $\{(1, 2), (3, 4), (5, 7), (7, 9), (9, 12)\}$

    (c) $\{(-2, 4), (-1, 3), (0, 1), (1, -1), (2, -3)\}$

20. In physics, Hooke's law states that (within certain limits) there is a linear relationship between the length x of a spring and the force y applied to (or exerted by) the spring. That is, $y = cx + d$, where c is called the **spring constant**. Use the following data to estimate the spring constant (the length is given in inches and the force is given in pounds).

Length x	Force y
3.5	1.0
4.0	2.2
4.5	2.8
5.0	4.3

21. Find the minimal solution to each of the following systems of linear equations.

    a) $x + 2y - z = 12$

    b) $x + 2y - z = 1$
    $2x + 3y + z = 2$
    $4x + 7y - z = 4$

    c) $x + y - z = 0$
    $2x - y + z = 3$
    $x - y + z = 2$

    d) $x + y + z - w = 1$
    $2x - y + w = 1$

22. Consider the problem of finding the least squares line $y = ct + d$ corresponding to the m observations $(t_1, y_1), (t_2, y_2), \ldots, (t_m, y_m)$.

    (a) We recall the result : Let $A \in M_{m\times n}(F)$ and $y \in F^m$. Then there exists $x_0 \in F^n$ such that $(A^*A)x_0 = A^*y$ and $\|Ax_0 - y\| \leq \|Ax - y\|$ for all $x \in F^n$. Furthermore, if $rank(A) = n$, then $x_0 = (A^*A)^{-1}A^*y$.

Show that the equation $(A^*A)x_0 = A^*y$ takes the form of the *normal equations:*

$$\left(\sum_{i=1}^{m} t_i^2\right) c + \left(\sum_{i=1}^{m} t_i\right) d = \sum_{i=1}^{m} t_i y_i$$

and

$$\left(\sum_{i=1}^{m} t_i\right) c + md = \sum_{i=1}^{m} y_i.$$

These equations may also be obtained from the error E by setting the partial derivatives of E with respect to both c and d equal to zero.

(b) Use the second normal equation of (a) to show that the least squares line must pass through the center of mass, $(\bar{t}, \bar{y})$, where

$$\bar{t} = \frac{1}{m}\sum_{i=1}^{m} t_i \quad \text{and} \quad \bar{y} = \frac{1}{m}\sum_{i=1}^{m} y_i.$$

23. Let V be the vector space of all sequences σ in F (where $F = \mathbb{R}$ or $F = \mathbb{C}$) such that $\sigma(n) \neq 0$ for only finitely many positive integers n. For $\sigma, \mu \in V$, we define $\langle \sigma, \mu \rangle = \sum_{n=1}^{\infty} \sigma(n)\overline{\mu(n)}$. Since all but a finite number of terms of the series are zero, the series converges. For each positive integer n, let e_n be the sequence defined by $e_n(k) = \delta_{n,k}$, where $\delta_{n,k}$ is the Kronecker delta. We proved that $\{e_1, e_2, \ldots\}$ is an orthonormal basis for V. Define $T : V \to V$ by

$$T(\sigma)(k) = \sum_{i=k}^{\infty} \sigma(i) \quad \text{for every positive integer k.}$$

Notice that the infinite series in the definition of T converges because $\sigma(i) \neq 0$ for only finitely many i.

(a) Prove that T is a linear operator on V.

(b) Prove that for any positive integer n, $T(e_n) = \sum_{i=1}^{n} e_i$.

(c) Prove that T has no adjoint.
Hint: By way of contradiction, suppose that T^* exists. Prove that for any positive integer n, $T^*(e_n)(k) \neq 0$ for infinitely many k.

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